Name: Farah Hjeeb

ID: 900077394

1

52/52

Assignment IV

* Question 1: Form a basis for Hom $(P_2, \mathbb{R}^{2\times 2})$

. $T_1' : R^2 \longrightarrow \mathbb{R}^4$ where:

. $T_1'(a_1, a_2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \sim T_1(a_1 + a_2 x) = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$

. $T_2'(a_1, a_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \sim T_2(a_1 + a_2 x) = \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}$

. $T_3'(a_1, a_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \\ 0 \\ 0 \end{bmatrix} \sim T_3(a_1 + a_2 x) = \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}$

. $T_4'(a_1, a_2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \sim T_4(a_1 + a_2 x) = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}$

. $T_5'(a_1, a_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a_1 \\ 0 \end{bmatrix} \sim T_5(a_1 + a_2 x) = \begin{pmatrix} 0 & 0 \\ a_1 & 0 \end{pmatrix}$

$\cdot\ T_6'(a_1, a_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a_2 \\ 0 \end{bmatrix} \sim T_6(a_1 + a_2 X) = \begin{pmatrix} 0 & 0 \\ a_2 & 0 \end{pmatrix}$

$\cdot\ T_7'(a_1, a_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \end{bmatrix} \sim T_7(a_1 + a_2 X) = \begin{pmatrix} 0 & 0 \\ 0 & a_1 \end{pmatrix}$

$\cdot\ T_8'(a_1, a_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_2 \end{bmatrix} \sim T_8(a_1 + a_2 X) = \begin{pmatrix} 0 & 0 \\ 0 & a_2 \end{pmatrix}$

So the basis of $\mathrm{Hom}(P_2, \mathbb{R}^{2\times 2}) = \mathrm{span}\{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$

(2)

* Question 2:

Let V be a vector space s.t $IN(V) = 8$

Let W and K be subspaces of V s.t $IN(W) = 5$ and $IN(K) = 4$

. We proved in class that $W + K$ is a subspace of V

then $IN(W+K) \leq IN(V)$

$$\Rightarrow IN(W) + IN(K) - IN(W \cap K) \leq IN(V)$$

$$9 - IN(W \cap K) \leq 8$$

$$IN(W \cap K) \geq 1$$

and $IN(W \cap K)$ can't be greater than (~~or equal to~~) $IN(W)$ and $IN(K)$

thus $IN(W \cap K) \leq 4$

so $1 \leq IN(W \cap K) \leq 4$

therefore $IN(W \cap K) = 1$ or 2 or 3 or 4

* Question 3: Let $T : P \to \mathbb{R}$ be a L.T 3

s.t $T(f(x)) = \left(\int_0^1 f(x)\,dx,\ f'(0),\ 0\right)$

So $T(a_1 + a_2 X + a_3 X^2 + a_4 X^3) = \left(a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \frac{1}{4}a_4,\ a_2,\ 0\right)$

a) first find the fake L.T $T' : \mathbb{R}^4 \to \mathbb{R}^3$ s.t:

$$T'(a_1, a_2, a_3, a_4) = \left(a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \frac{1}{4}a_4,\ a_2,\ 0\right)$$

so the standard matrix presentation is:

$$M = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

b) $\text{Range}(T') = \left\{\left(a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \frac{1}{4}a_4,\ a_2,\ 0\right) \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\right\}$

$$= \left\{a_1(1,0,0) + a_2\left(\frac{1}{2},1,0\right) + a_3\left(\frac{1}{3},0,0\right) + a_4\left(\frac{1}{4},0,0\right) \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\right\}$$

$$= \text{span}\left\{(1,0,0), \left(\frac{1}{2},1,0\right)\right\}$$

and $\text{Range}(T) = \text{span}\left\{(1,0,0), \left(\frac{1}{2},1,0\right)\right\}$

c) To find $Z(T)$, first we will solve the system $M\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} a_2 = 0 \\ a_1 = -\frac{1}{3}a_3 - \frac{1}{4}a_4 \end{cases}$$

so $Z(T') = \left\{\left(-\frac{1}{3}a_3 - \frac{1}{4}a_4,\ 0,\ a_3,\ a_4\right) \mid a_3, a_4 \in \mathbb{R}\right\}$

$$= \text{span}\left\{\left(-\frac{1}{3},0,1,0\right), \left(-\frac{1}{4},0,0,1\right)\right\}$$

thus $Z(T) = \text{span}\left\{\left(-\frac{1}{3} + X^2\right), \left(-\frac{1}{4} + X^3\right)\right\}$

* Question 4: $T: \mathbb{R}^4 \to \mathbb{R}^4$ s.t $T(a_1, a_2, a_3, a_4) = (2a_1 + a_3, 0, a_1, a_1)$

and $F: \mathbb{R}^4 \to \mathbb{R}^4$ s.t $F(b_1, b_2, b_3, b_4) = (b_1 + b_2, -3b_1 - 3b_2, b_3, 4b_3)$

a) $T+F: \mathbb{R}^4 \to \mathbb{R}^4$ is a L.T s.t:

$$(T+F)(c_1, c_2, c_3, c_4) = T(c_1, c_2, c_3, c_4) + F(c_1, c_2, c_3, c_4)$$
$$= (2c_1 + c_3, 0, c_1, c_1) + (c_1 + c_2, -3c_1 - 3c_2, c_3, 4c_3)$$
$$= (3c_1 + c_2 + c_3, -3c_1 - 3c_2, c_1 + c_3, c_1 + 4c_3)$$

and $M_T = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $M_F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -3 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix}$

thus $M_{T+F} = M_T + M_F = \begin{pmatrix} 3 & 1 & 1 & 0 \\ -3 & -3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 \end{pmatrix}$

$\frac{4}{4}$

b) $\text{Range}(T+F) = \text{span}\{(3, -3, 1, 1), (1, -3, 0, 0), (1, 0, 1, 4)\}$

c) To find $Z(T+F)$ we must solve the system $M_{T+F} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

thus $\left(\begin{array}{cccc|c} 3 & 1 & 1 & 0 & 0 \\ -3 & -3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 \end{array}\right) \xrightarrow[-R_3 + R_4 \to R_4]{R_1 + R_2 \to R_2} \left(\begin{array}{cccc|c} 3 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right)$

Thus $c_3 = c_2 = c_1 = 0$

$\longrightarrow$

so $Z(T+F) = \{(0,0,0,c_4) \mid c_4 \in \mathbb{R}\}$

$= \{c_4(0,0,0,1) \mid c_4 \in \mathbb{R}\}$

$= \text{Span}\{(0,0,0,1)\}$

$\frac{4}{4}$

d) we know that $M_{T^2} = (M_T)^2 = M_T \cdot M_T$

thus $M_{T^2} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$

$\frac{4}{4}$

e) $\text{Range}(T^2) = \text{span}\{(5,0,2,2), (2,0,1,1)\}$

$\frac{4}{4}$

5

* Question 5:

a) Let A be an idempotent matrix i.e $A^2 = A$.

case 1) $|A| \neq 0$

$\Leftrightarrow A$ is invertible

$\Leftrightarrow \exists A^{-1}$ s.t $A^2 = A \Rightarrow \underbrace{A^{-1}A}A = A^{-1}A$

$\Rightarrow A = I_n$.

OK I see it

case 2) suppose $A \neq I_n$ then $|A| = 0$

why! i.e., why I_n is the only invertible idempotent ?

$\Rightarrow A$ is non-invertible

$\Rightarrow \exists X \in \mathbb{R}^n$ s.t $AX = 0$ and $X \neq 0$

X $\Rightarrow$ the system has infinitely many solutions since every scalar multiplication of X is a solution to the system.

b) $(I_n - A)^2 = I_n^2 - 2I_nA + A^2$

$= I_n - 2A + A$

$= I_n - A$.

therefore $(I_n - A)$ is idempotent matrix.

~~Proof (Trivial). Suppose A is invertible and $A^2 = A$. Hence $A^{-1}A^2 = A^{-1}A$. Thus $A^{-1}AA = A^{-1}A = I_n$. Hence $A = I_n$.~~

c) Let A, $n \times n$, be a nilpotent matrix i.e $A^m = 0$

suppose that $|A + I_n| = 0$

then $(A + I_n) V = 0$ for some $V \neq 0 \in \mathbb{R}^n$

$$\Rightarrow \quad AV = (-1) V$$

$$\Rightarrow \quad A^m V = (-1)^m V$$

$\Rightarrow \quad A^m = (-1)^m I_n$ which contradicts $A^m = 0$

therefore $|A + I_n| \neq 0$

which implies that $A + I_n$ is invertible.

4/4